one exceptional set of curves, obtained at the United States Model Basin,* which shows two maxima: a phenomenon which has not received explanation. It is conceivable that this may be a case in which the two maxima indicated in the intermediate curves of the present paper have become prominent through some unusual features of the model. In this connection, it must be remembered that the present calculations are based upon a surface pressure of specially simple type, one symmetrical round a point; one could extend the calculations by integration, as in the previous results for deep water, so as to apply to a pressure distribution, giving a better analogy with ship form. It may be anticipated that the results would be of the same character in general, though no doubt better agreement could be obtained in certain details.

The Diffraction of Plane Electromagnetic Waves by a Perfectly Reflecting Sphere.

By F. Puryer White, M.A., Fellow of St. John's College, Cambridge.

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Introduction.—The problem of the diffraction of electromagnetic waves by a perfectly conducting sphere is of interest both from the point of view of wireless and from that of physical optics. Two cases may be considered: (1) when the source of the waves is a Hertzian oscillator on the surface of the sphere; and (2) when the waves are plane. The formal series solution of both these problems has been given by several writers, including Sir J. J. Thomson, the late Lord Rayleigh, and Prof. H. M. Macdonald. For a sphere of which the radius is small compared with the wave-length the series converge rapidly and are suitable for computation, but for a large sphere the important terms are far on in the series and the latter must be transformed in order to get formulæ which may be of use.

For case (1) this problem has been attacked by Macdonald, Poincaré, J. W. Nicholson, A. E. H. Love and several others, but case (2) has not attracted nearly so much attention. This paper, then, deals with case (2).

* D. W. Taylor, *loc. cit.*, p. 115.

The most recent paper on the subject is by Dr. Bromwich,* who gives the series solution in the following form :—

At a large distance, r, from the sphere, of radius a, the scattered wave is given by components in the directions of r, θ, ϕ respectively

$$X = c\alpha = 0,$$

$$Y = c\gamma = \frac{\partial M}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial N}{\partial \phi},$$

$$Z = -c\beta = \frac{1}{\sin\theta}\frac{\partial M}{\partial \phi} + \frac{\partial N}{\partial \theta},$$

where

$$M = -\sin\theta\cos\phi\,\frac{e^{-ikr}}{kr}\sum_1^\infty(-1)^n\,\frac{2n+1}{n(n+1)}\,\frac{S_n'(ka)}{E_n'(ka)}\,P_n'(\cos\theta),$$

$$N = -\sin\theta\sin\phi\,\frac{e^{-ikr}}{kr}\sum_1^\infty(-1)^n\,\frac{2n+1}{n(n+1)}\,\frac{S_n(ka)}{E_n(ka)}\,P_n'(\cos\theta). \qquad (1)$$

The incident wave is of unit amplitude, is travelling along the axis $\theta = 0$, is polarised in the plane $\phi = \frac{1}{2}\pi$, and has wave-length $2\pi/k$. The time-factor e^{ikct} is omitted throughout.

$S_n(ka)$, $E_n(ka)$ are the Riccati-Bessel functions which have been tabulated for integral values of n and ka up to 10 by the British Association Committee.†

Dr. Bromwich (*loc. cit.*) has transformed the series M and N, for ka large, by an application of Kelvin's "Method of stationary phase," and gets the approximate formulæ

$$Y = \cos\phi\,(e^{-ikr}/kr)\,\tfrac{1}{2}ka\exp(2ika\cos\tfrac{1}{2}\theta),$$

$$Z = -\sin\phi\,(e^{-ikr}/kr)\,\tfrac{1}{2}ka\exp(2ika\cos\tfrac{1}{2}\theta). \qquad (2)$$

These formulæ, the equivalents of which were published in the earlier papers of Nicholson, are for points not near to the axis $\theta = \pi$ behind the sphere; the method does not seem to be capable of dealing with points near to this axis, nor does it seem suitable for obtaining a second approximation.

For points near to the axis $\theta = \pi$, Dr. Bromwich has applied another method, depending on the use of Green's theorem, and obtains

$$Y = iY_2\cos\phi\,(e^{-ikr}/kr), \qquad Z = iZ_2\sin\phi\,(e^{-ikr}/kr), \qquad (3)$$

where $Y_2 = Z_2 = -kaJ_1(ka\sin\theta)/\sin\theta$.

These points are not dealt with by Nicholson.

* Bromwich, 'Phil. Trans.,' A, vol. 220, p. 187 (1920). See also Nicholson, 'Proc. London Math. Soc.,' vol. 9, p. 67 (1910); vol. 11, p. 277 (1912).

† Doodson, etc., 'B. A. Report,' p. 13 (1914); p. 39 (1916).

With a view to verification of these formulæ, numerical results* have been calculated directly from the original series for $ka = 10$. Formulæ (2) were found to give quite good agreement for the range $\theta = 0°$ to $90°$; for the remainder of the range there was marked divergence. Formulæ (3) again give fair agreement for the range 170°–180°.

A paper by Prof. G. N. Watson† on the problem of case (1), mentioned above, suggested another method of attack. Watson transforms his series into another depending on the zeros of a Bessel function, and shows that this latter series converges very rapidly for large values of the argument, the first term alone giving a sufficient approximation.

It very soon appeared, as indeed might have been anticipated from the form of Dr. Bromwich's approximation, that it was not possible to transform the series M and N into others in Watson's manner, the contribution to the contour integrals from a large semicircle not tending to zero as the radius tends to infinity.

An example in Whittaker's 'Modern Analysis'‡ suggested a rather different procedure, whereby the series M and N are transformed into series depending on the zeros of Bessel functions, together with a contour integral along a "path of steepest descent." The contribution from the series is very small, and thus an approximation is given by the contour integral. This can be evaluated, after the manner of Debye,§ as an asymptotic series; the first term gives Bromwich's formulæ (2), and a second approximation can be obtained.

Comparison with the figures of Proudman shows an improved agreement over the range 0°–90°, but no better agreement over the remainder of the range.

The method breaks down as θ approaches π for two reasons:

(1) Laplace's approximation for the Legendre function $P_s(-\cos\theta)$ of high order s is no longer valid when θ is so close to π that $s(\pi-\theta)^3$ is small.

(2) Debye's simple approximations for Bessel functions of large argument are no longer valid when the order and argument are nearly the same.

The first fact shows itself in the series which we neglect for θ not near to π; this is found to involve exponentials with negative indices $Ax^{\frac{1}{3}}(\theta-\pi)$, A being a positive constant, and the terms will no longer be negligible when $x^{\frac{1}{3}}(\pi-\theta)$ is small (x is written for ka).

This difficulty can be evaded by the use of Mehler's approximation for

* Proudman, Doodson and Kennedy, 'Phil. Trans.,' A, vol. 217, p. 279 (1917).

† Watson, 'Roy. Soc. Proc.,' A, vol. 95, p. 83 (1918).

‡ 2nd edition, p. 145, Example 7.

§ Debye, 'Math. Annalen,' vol. 67, p. 535 (1909).

the Legendre function in terms of a Bessel function of zero order, and we find that the series M and N are now transformed into

(*a*) A series depending on the roots of a Bessel function, this series being still negligibly small;

(*b*) Several comparatively simple contour integrals which are negligible in the earlier case, but important now—approximations to these can easily be obtained;

(*c*) A contour integral involving Bessel functions of order s and argument x along a curve in the s-plane which passes through the point $s = x$.

The evaluation of (*c*) is a matter of considerable complication and labour. In the first place, approximations to the various Bessel functions for nearly equal values of order and argument are required; some of these are given by Watson,* but others had to be worked out on similar lines.

These approximations involve Bessel functions of orders $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, and the leading terms in (*c*) are found to depend upon definite numbers arising in the form of integrals of which the integrands are complicated expressions in these. To calculate these numbers, Dinnik's† Tables of the Bessel functions involved are of use, but they contain several misprints, and are not sufficiently detailed for small values of the argument, so that a considerable amount of arithmetic has been necessary.

The final result, for θ near to π, is complicated; there is a term of order x^2, which is like Dr. Bromwich's but not identical with it, and there are further terms in $x^{4/3}$. Much better agreement is found with Proudman's figures for the range 170°–180°, but again the range of validity is little increased. For larger values of ka, the next terms in the approximation, involving presumably $x^{2/3}$, would be important; the labour necessary to obtain these by this method would be very great.

The intermediate range, 90°–170°, is thus still without adequate treatment.

(1) The series to which approximations are required are

$$M = -\sin\theta\cos\phi\,\frac{e^{-ikr}}{kr}\sum_1^\infty(-1)^n\,\frac{2n+1}{n(n+1)}\,\frac{S_n'(ka)}{E_n'(ka)}\,P_n'(\cos\theta),$$

$$N = -\sin\theta\sin\phi\,\frac{e^{-ikr}}{kr}\sum_1^\infty(-1)^n\,\frac{2n+1}{n(n+1)}\,\frac{S_n(ka)}{E_n(ka)}\,P_n'(\cos\theta).$$

With the notation of Watson,‡

$$S_n(x) = \psi_n(x) = \tfrac{1}{2}\{\eta_n(x)+\zeta_n(x)\},$$
$$E_n(x) = -i\zeta_n(x),$$

where $\quad \eta_n(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}}H_{n+\frac{1}{2}}^{(1)}(x), \qquad \zeta_n(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}}H_{n+\frac{1}{2}}^{(2)}(x),$

* Watson, 'Proc. Camb. Phil. Soc.,' vol. 19, p. 96 (1916).

† A. Dinnik, 'Archiv der Math. und Phys.' (3), vol. 22, p. 226 (1914).

‡ Watson, 'Roy. Soc. Proc.,' A, vol. 95, p. 83 (1918).

and thus, since when n is a positive integer

$$P_n'(-\cos\theta) = (-1)^{n+1} P_n'(\cos\theta),$$

we have $\quad M = iA\cos\phi\,(e^{-ikr}/kr), \qquad N = iB\sin\phi\,(e^{-ikr}/kr),$

where
$$A = \sum_1^\infty \frac{2n+1}{n(n+1)} P_n'(-\cos\theta)\sin\theta\,\frac{\psi_n'(x)}{\zeta_n'(x)},$$
$$B = \sum_1^\infty \frac{2n+1}{n(n+1)} P_n'(-\cos\theta)\sin\theta\,\frac{\psi_n(x)}{\zeta_n(x)},$$

and x is written for ka.

(2) Following the example of Prof. G. N. Watson,* we shall consider Bessel functions of which the order s is a complex quantity, and we shall use the transformation

$$s = x\cosh\gamma = x\cosh(\alpha + i\beta),$$

whereby the s-plane, slit along the real axis except between the points $\pm x$, is represented by the strip of the γ-plane between $\beta = 0$ and $\beta = \pi$.

Fig. 1 represents the s-plane divided by the curves into regions

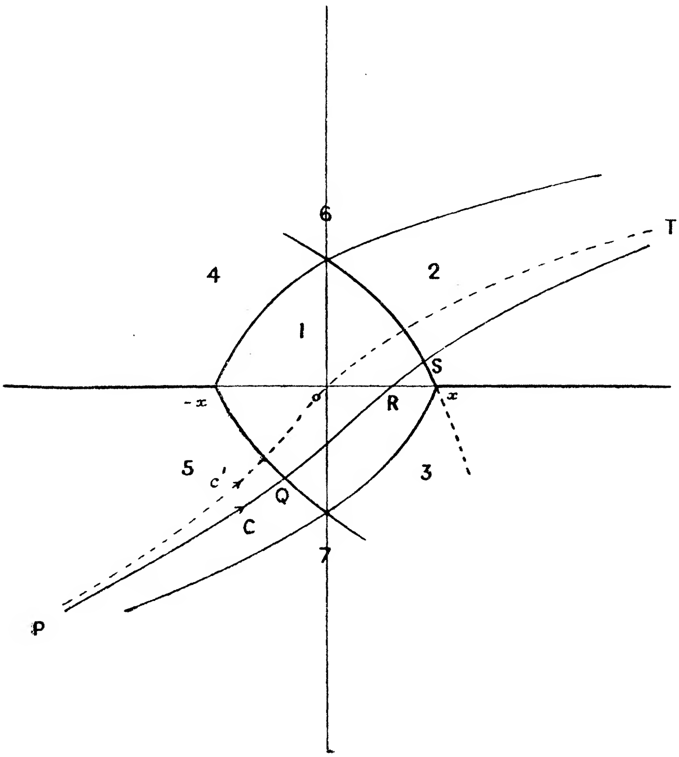

Fig. 1.

* Watson, *loc. cit.*

(numbered 1–7) in which the Hankel-Bessel functions are given by different contour integrals, as shown in the Table on p. 93 of Watson's paper.

We shall require to apply the method of "steepest descents" to integrals involving

$$\exp\left[2x\{\sinh\gamma - \gamma\cosh\gamma + \tfrac{1}{2}i(\pi-\theta)\cosh\gamma\}\right].$$

Suitable curves must pass through the point where this index is stationary, *i.e.*, the point $\gamma = \frac{1}{2}(\pi-\theta)i$, and must have

$$\mathrm{I}\{\sinh\gamma - \gamma\cosh\gamma + \tfrac{1}{2}i(\pi-\theta)\cosh\gamma\} = \text{const.} = \cos\tfrac{1}{2}\theta.$$

This gives

$$\cosh\alpha\sin\beta - \alpha\sinh\alpha\sin\beta + \{\tfrac{1}{2}(\pi-\theta)-\beta\}\cosh\alpha\cos\beta = \cos\tfrac{1}{2}\theta.$$

There is a double point at $\alpha = 0$, $\beta = \frac{1}{2}(\pi-\theta)$, the branches there having tangents $\alpha = \pm\{\beta - \frac{1}{2}(\pi-\theta)\}$.

Along the branch touching $\alpha = -\beta + \frac{1}{2}(\pi-\theta)$, the real part of

$$\sinh\gamma + \{\tfrac{1}{2}(\pi-\theta)i - \gamma\}\cosh\gamma$$

is negative; this branch goes off to infinity at $(\infty, 0)$ and $(-\infty, \pi)$.

The corresponding curve, which we will call C, in the s-plane, passes

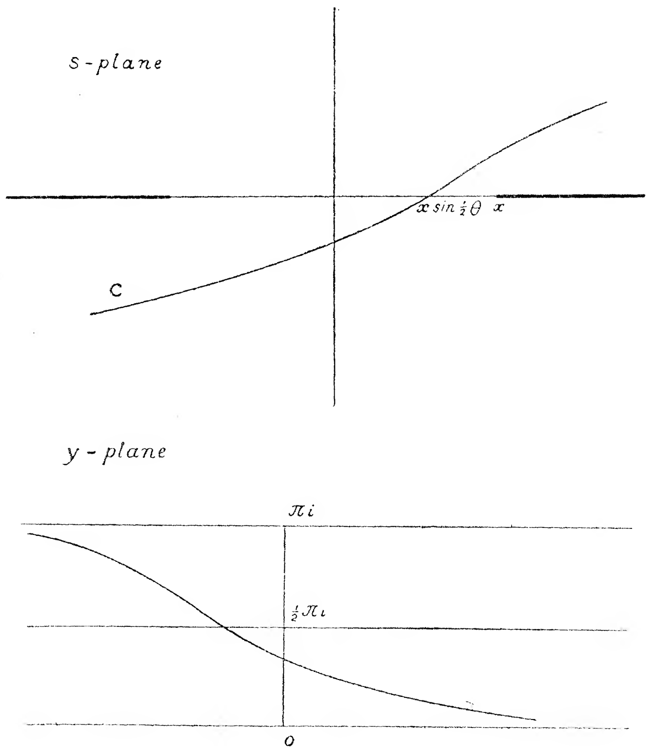

FIG. 2.

through the point $s = x \sin \frac{1}{2}\theta$ of the real axis and goes off to infinity in directions parallel to the real axis but not asymptotic to it. These curves are shown in fig. 2.

(3) Now consider the contour integral

$$\int \frac{2s}{s^2-\frac{1}{4}} \sin\theta \, P_{s-\frac{1}{2}}'(-\cos\theta) \frac{\psi_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)} \frac{1}{e^{2s\pi i}+1} ds, \tag{3·1}$$

taken round a circuit in the s-plane consisting of any curve C′ symmetrical with regard to the origin and passing through it from the third quadrant to the first, together with a large semicircle below C′. The curve C′ must not cross the boundary between the regions 6 and 2 and between 5 and 7; it must, in fact, behave at infinity like C.

Now in Watson's regions 5, 7, 3, 2 we have

$$H_s^{(1)}(x) = S_s^{(1)}(x) - e^{-2s\pi i} S_s^{(2)}(x), \quad \frac{S_s^{(1)}(x) - e^{-2s\pi i} S_s^{(2)}(x)}{1-e^{-2s\pi i}},$$

$$S_s^{(1)}(x), \quad S_s^{(1)}(x) - S_s^{(2)}(x);$$

$$H_s^{(2)}(x) = S_s^{(2)}(x), \quad \frac{S_s^{(2)}(x) - S_s^{(1)}(x)}{1-e^{-2s\pi i}}, \quad S_s^{(2)}(x) - S_s^{(1)}(x), \quad S_s^{(2)}(x),$$

respectively, where when $|\gamma|$ is not small and x is large,

$$S_s^{(1)}(x) \sim e^{x(\sinh\gamma - \gamma\cosh\gamma) - \frac{1}{4}\pi i} \div \sqrt{\{\tfrac{1}{2}\pi x \sin(-i\gamma)\}},$$

$$S_s^{(2)}(x) \sim e^{-x(\sinh\gamma - \gamma\cosh\gamma) + \frac{1}{4}\pi i} \div \sqrt{\{\tfrac{1}{2}\pi x \sin(-i\gamma)\}}.$$

Thus
$$\frac{2\psi_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)} = 1 - e^{-2s\pi i} + \frac{S_s^{(1)}(x)}{S_s^{(2)}(x)}, \quad \frac{(1-e^{-2s\pi i}) S_s^{(2)}(x)}{S_s^{(2)}(x) - S_s^{(1)}(x)},$$

$$\frac{S_s^{(2)}(x)}{S_s^{(2)}(x) - S_s^{(1)}(x)}, \quad \frac{S_s^{(1)}(x)}{S_s^{(2)}(x)},$$

respectively, and

$$|(\tfrac{1}{2}\pi x)^{\frac{1}{2}} S_s^{(1)}(x)| \sim |\sinh\gamma|^{-\frac{1}{2}} e^{P},$$

$$|(\tfrac{1}{2}\pi x)^{\frac{1}{2}} S_s^{(2)}(x)| \sim |\sinh\gamma|^{-\frac{1}{2}} e^{-P},$$

where P is the real part of $x(\sinh\gamma - \gamma\cosh\gamma)$.

If $s = Re^{\mp i\phi}$, ϕ being a positive angle not greater than $\frac{1}{2}\pi$, we have when R is very large

$$|\sinh\gamma| \sim R/x,$$

$$P \sim R[\cos\phi(\log 2R/x - 1) - \phi\sin\phi].$$

Thus in 5, P is negative and of order $R\cos\phi \log 2R/x$.

As we move round the semicircle through 7 to 3 and 2, P increases, taking the value $-\frac{1}{2}R\pi$ on the imaginary axis, vanishing on entering the fourth quadrant, and becoming positive in the remainder of 7 and in 3 with the approximate value $R\cos\phi \log 2R/x$. In 2, P is negative like $-R\cos\phi \log 2R/x$.

Thus $\left|\dfrac{\psi_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)}\right|$ is of order 1 on the semi-circle, except in the latter part of 7 and in regions 3 and 2, where it is of order $e^{-2|\mathrm{P}|}$.

Also the approximate value of $\mathrm{P}_{s-\frac{1}{2}}'(-\cos\theta)\sin\theta$ for large values of s is $(2s/\pi\sin\theta)^{\frac{1}{2}}\sin\{s(\pi-\theta)-\frac{1}{4}\pi\}$, when θ is not near to 0 or π.

Thus approximately $|\mathrm{P}_{s-\frac{1}{2}}'(-\cos\theta)\sin\theta| \sim (\mathrm{R}/2\pi\sin\theta)^{\frac{1}{2}}e^{\mathrm{R}(\pi-\theta)\sin\phi}$. Hence the modulus of the integrand behaves like

$(2\mathrm{R}/\pi\sin\theta)^{\frac{1}{2}}e^{-\mathrm{R}(\pi+\theta)\sin\phi}$ in region 5 and the first part of 7,

like $(2\mathrm{R}/\pi\sin\theta)^{\frac{1}{2}}e^{-\mathrm{R}(\pi+\theta)\sin\phi-2|\mathrm{P}|}$ in the remainder of 7 and in 3,

and like $(2\mathrm{R}/\pi\sin\theta)^{\frac{1}{2}}e^{-2|\mathrm{P}|}$ in region 2.

Thus the integral round the large semi-circle tends to zero as its radius tends to infinity.

Next consider the poles of the integrand which lie within the contour. These consist of the zeros of $\zeta_{s-\frac{1}{2}}(x)$, and of the points $s=n+\frac{1}{2}$, where n is zero or a positive integer.

The residue at $s=n+\frac{1}{2}\ (n>0)$ is $-\dfrac{1}{2\pi i}\dfrac{2n+1}{n(n+1)}\sin\theta\,\mathrm{P}_n'(-\cos\theta)\dfrac{\psi_n(x)}{\zeta_n(x)}$.

The residue at $s=\frac{1}{2}$ is $\dfrac{1}{2\pi}\sin x e^{ix}\cot\frac{1}{2}\theta$.

As regards the zeros of $\zeta_{s-\frac{1}{2}}(x)$, Watson* has shown that these are simple and not on the real axis, and in fact lie near to the dotted curve proceeding from the point $s=x$ into the fourth quadrant.

The residue at a zero $s=\nu$ is

$$\frac{\nu}{\nu^2-\frac{1}{4}}\sin\theta\,\mathrm{P}_{\nu-\frac{1}{2}}'(-\cos\theta)\frac{\eta_{\nu-\frac{1}{2}}(x)}{[\partial/\partial s\cdot\zeta_{s-\frac{1}{2}}(x)]_{s=\nu}}\frac{1}{e^{2\nu\pi i}+1}.$$

Thus we get, after some re-arrangement,

$$\begin{aligned}\mathrm{B} = {} & -i\sin x e^{ix}\cot\tfrac{1}{2}\theta \\ & +2\pi i\sum_{\nu}\frac{\nu}{\nu^2-\frac{1}{4}}\sin\theta\,\mathrm{P}_{\nu-\frac{1}{2}}'(-\cos\theta)\frac{\eta_{\nu-\frac{1}{2}}(x)}{[\partial/\partial s\,\zeta_{s-\frac{1}{2}}(x)]_{s=\nu}}\frac{1}{e^{2\nu\pi i}+1} \\ & +\int_{\mathrm{C}'}\frac{2s}{s^2-\frac{1}{4}}\sin\theta\,.\,\mathrm{P}_{s-\frac{1}{2}}'(-\cos\theta)\,.\,\frac{\psi_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)}\frac{1}{e^{2s\pi i}+1}\,ds. \qquad (3\cdot2)\end{aligned}$$

(4) Next consider the contour integral

$$\int_{\mathrm{C}'}\frac{s}{s^2-\frac{1}{4}}\sin\theta\,\mathrm{P}_{s-\frac{1}{2}}'(-\cos\theta)\frac{\eta_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)}\frac{1}{e^{-2s\pi i}+1}\,ds, \qquad (4\cdot1)$$

taken along the curve C′.

In region 5, $\left|\dfrac{\eta_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)}\right| \sim e^{-2\mathrm{R}\pi\sin\phi}$; in region 2, $\left|\dfrac{\eta_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)}\right| \sim 1$, and thus

* Watson, *loc. cit.*, § 5.

the modulus of the integrand at each end of the curve behaves like $(R/2\pi \sin\theta)^{\frac{1}{2}} e^{-R(\pi+\theta)\sin\phi}$ and the integral converges.

Now we have $H_{-s}^{(1)}(x) = e^{s\pi i} H_s^{(1)}(x)$, $H_{-s}^{(2)}(x) = e^{-s\pi i} H_s^{(2)}(x)$, and thus $\frac{\eta_{-s-\frac{1}{2}}(x)}{\zeta_{-s-\frac{1}{2}}(x)} \frac{1}{e^{2s\pi i}+1} = \frac{\eta_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)} \frac{1}{e^{-2s\pi i}+1}$, *i.e.*, this expression is an even function of s. So also is $P_{s-\frac{1}{2}}'(-\cos\theta)$.

Hence the integrand is an odd function of s and the integral vanishes.

Adding to the equation (3·2), we get

$$
\begin{aligned}
B = & -i \sin x e^{ix} \cot \tfrac{1}{2}\theta \\
& + 2\pi i \sum_\nu \nu/(\nu^2-\tfrac{1}{4}) \sin\theta \, P_{\nu-\frac{1}{2}}'(-\cos\theta) \frac{\eta_{\nu-\frac{1}{2}}(x)}{(e^{2\nu\pi i}+1)[\partial/\partial s\, \zeta_{s-\frac{1}{2}}(x)]_{s=\nu}} \\
& + \int_{C'} \frac{s}{s^2-\frac{1}{4}} \sin\theta \,.\, P_{s-\frac{1}{2}}'(-\cos\theta) \left[\frac{2\psi_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)} \frac{1}{e^{2s\pi i}+1} + \frac{\eta_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)} \frac{1}{e^{-2s\pi i}+1}\right] ds.
\end{aligned} \tag{4·2}
$$

The integral in equation (4·2) can be written

$$
I_2 = \int_{C'} \frac{s}{s^2-\frac{1}{4}} \sin\theta \, P_{s-\frac{1}{2}}'(-\cos\theta) \left[\frac{2\psi_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)} - \frac{1}{e^{-2s\pi i}+1}\right] ds \tag{4·3}
$$

$$
= \int_{C'} \frac{s}{s^2-\frac{1}{4}} \sin\theta \, P_{s-\frac{1}{2}}'(-\cos\theta) \left[\frac{\eta_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)} + \frac{1}{e^{2s\pi i}+1}\right] ds. \tag{4·4}
$$

The form (4·3) is suitable on the upper half of the curve, each part converging at this end; the form (4·4) is similarly suitable on the lower half.

Equation (4·2) has been proved for θ not equal to 0 or π; it can easily be seen that it holds for $\theta = \pi$, but it does not hold for $\theta = 0$, as the Legendre functions of non-integral order become infinite in this case.

(5) As regards the integral I_2, we note that $\sin\theta \, P_{s-\frac{1}{2}}'(-\cos\theta)$ can be expressed as the sum of two terms

$$
e^{[s(\pi-\theta)-\frac{1}{4}\pi]i} C_1 + e^{[-s(\pi-\theta)+\frac{1}{4}\pi]i} C_2,
$$

where C_1 and C_2 can be expressed either as definite integrals or as asymptotic series in s^{-1} not involving exponentials.*

Now consider the integral

$$
I_2' = \int \frac{2s}{s^2-\frac{1}{4}} e^{[-s(\pi-\theta)+\frac{1}{4}\pi]i} C_2 \,.\, \frac{\psi_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)} ds,
$$

the circuit consisting of C' and the large semi-circle below C'.

On the part of the semi-circle below the real axis, the integral tends to zero on account of the factor $e^{-s(\pi-\theta)i}$; on the part above the real axis (in region 2) it tends to zero on account of the factor $\frac{\psi_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)}$.

* Watson, 'Camb. Phil. Trans.,' vol. 22, pp. 291, 295 (1918).

Thus $$I_2' = -2\pi i \sum_{\nu} \frac{\nu}{\nu^2-\frac{1}{4}} e^{[-\nu(\pi-\theta)+\frac{1}{4}\pi]i} C_{2(\nu)} \frac{\eta_{\nu-\frac{1}{2}}(x)}{[\partial/\partial s\, \zeta_{s-\frac{1}{2}}(x)]_{s=\nu}}. \tag{5·1}$$

The remaining part of I_2 is

$$I_2'' = \int_{C'} \frac{s}{s^2-\frac{1}{4}} \left\{ \frac{2\psi_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)} e^{[s(\pi-\theta)-\frac{1}{4}\pi]}C_1 - \frac{\sin\theta\, P_{s-\frac{1}{2}}'(-\cos\theta)}{e^{-2s\pi i}+1} \right\} ds. \tag{5·2}$$

To get this in a form involving an integral along a "curve of steepest descent," we must consider the appropriate approximations for $\psi_{s-\frac{1}{2}}(x)$ and $\zeta_{s-\frac{1}{2}}(x)$ at various points of the curves C and C′. These curves pass through Watson's regions 5, 1 and 2, in all of which $\zeta_{s-\frac{1}{2}}(x)$ is represented by $(\frac{1}{2}\pi x)^{\frac{1}{2}} S_s^{(2)}(x)$; but in which $\eta_{s-\frac{1}{2}}(x)$ is represented respectively by $(\frac{1}{2}\pi x)^{\frac{1}{2}}\{S_s^{(1)}(x)-e^{-2s\pi i} S_s^{(2)}(x)\}$, $(\frac{1}{2}\pi x)^{\frac{1}{2}} S_s^{(1)}(x)$ and $(\frac{1}{2}\pi x)^{\frac{1}{2}}\{S_s^{(1)}(x)-S_s^{(2)}(x)\}$.

Hence, as the integrand for which C is a suitable curve must contain the factor $S_s^{(1)}(x)/S_s^{(2)}(x)$, which of course is discontinuous at the points Q, S, where C crosses the boundary curves between the various regions, the integral involved must be (see fig. 1)

$$\int_P^Q \frac{s}{s^2-\frac{1}{4}} e^{[s(\pi-\theta)-\frac{1}{4}\pi]i} C_1 \left[\frac{\eta_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)} + e^{-2s\pi i}\right] ds + \int_Q^S \frac{s}{s^2-\frac{1}{4}} e^{[s(\pi-\theta)-\frac{1}{4}\pi]i} C_1 \frac{\eta_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)} ds$$
$$+ \int_S^T \frac{s}{s^2-\frac{1}{4}} e^{[s(\pi-\theta)-\frac{1}{4}\pi]i} C_1 \left[\frac{\eta_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)} + 1\right] ds.$$

The remaining part of I_2'' is, on reduction,

$$-\int_P^Q \frac{s}{s^2-\frac{1}{4}} e^{-[s(\pi+\theta)+\frac{1}{4}\pi]i} C_1 ds - \int_S^T \frac{s}{s^2-\frac{1}{4}} e^{[s(\pi-\theta)-\frac{1}{4}\pi]i} C_1\, ds$$
$$+ \int_{C'} \frac{s}{s^2-\frac{1}{4}} \left[\frac{e^{[s(\pi-\theta)-\frac{1}{4}\pi]i}}{e^{2s\pi i}+1} C_1 - \frac{e^{-[s(\pi-\theta)-\frac{1}{4}\pi]i}}{e^{-2s\pi i}+1} C_2 \right] ds.$$

This last integral can easily be seen to vanish over the large semi-circle below C′, and thus the contribution of this last term is

$$\sum_1^\infty \frac{n+\frac{1}{2}}{n(n+1)} \sin\theta\, P_n'(-\cos\theta) - \frac{1}{2}\cot\frac{1}{2}\theta.$$

The series can easily be evaluated; its sum is $-\frac{1}{2}\tan\frac{1}{2}\theta$. This contribution is therefore equal to $-\operatorname{cosec}\theta$.

Thus

$$I_2'' = -\operatorname{cosec}\theta - \int_P^Q \frac{s}{s^2-\frac{1}{4}} e^{-[s(\pi+\theta)+\frac{1}{4}\pi]i} C_1\, ds - \int_S^T \frac{s}{s^2-\frac{1}{4}} e^{[s(\pi-\theta)-\frac{1}{4}\pi]i} C_1\, ds$$
$$+ \int_C \frac{s}{s^2-\frac{1}{4}} \frac{S_s^{(1)}(x)}{S_s^{(2)}(x)} \cdot e^{[s(\pi-\theta)-\frac{1}{4}\pi]i} C_1\, ds$$
$$- 2\pi i \sum_{\nu} \frac{\nu}{\nu^2-\frac{1}{4}} e^{[-\nu(\pi-\theta)+\frac{1}{4}\pi]i} C_{2(\nu)} \frac{\eta_{\nu-\frac{1}{2}}(x)}{[\partial/\partial s\, \zeta_{s-\frac{1}{2}}(x)]_{s=\nu}}. \tag{5·3}$$

This is for θ not equal to 0 or π.

(6) Collecting the results so far obtained we get

$$\mathrm{B} = -\tfrac{1}{2}\tan\tfrac{1}{2}\theta - \tfrac{1}{2}e^{2ix}\cot\tfrac{1}{2}\theta + 2\pi i \sum_{\nu} \nu/(\nu^2-\tfrac{1}{4}) \,.\, \eta_{\nu-\frac{1}{2}}(x)/[\partial/\partial s\, \zeta_{s-\frac{1}{2}}(x)]_{s=\nu}\,.$$

$$\left[\frac{e^{[\nu(\pi-\theta)-\frac{1}{4}\pi]i}}{e^{2s\pi i}+1}\mathrm{C}_1 - \frac{e^{-[\nu(\pi-\theta)-\frac{1}{4}\pi]i}}{e^{-2s\pi i}+1}\mathrm{C}^2\right] - \int_{\mathrm{P}}^{\mathrm{Q}} \frac{s}{s^2-\frac{1}{4}}\, e^{-[s(\pi+\theta)+\frac{1}{4}\pi]i}\,\mathrm{C}_1\, ds$$

$$-\int_{\mathrm{S}}^{\mathrm{T}} \frac{s}{s^2-\frac{1}{4}}\, e^{[s(\pi-\theta)-\frac{1}{4}\pi]i}\,\mathrm{C}_1\, ds + \int_{\mathrm{C}} \frac{s}{s^2-\frac{1}{4}}\,\frac{\mathrm{S}_s{}^{(1)}(x)}{\mathrm{S}_s{}^{(2)}(x)}\, e^{[s(\pi-\theta)-\frac{1}{4}\pi]i}\,\mathrm{C}_1\, ds. \quad (6{\cdot}1)$$

Precisely similar work applies to the series A; the only difference being that we have $\eta_{s-\frac{1}{2}}{}'(x)$ and $\zeta_{s-\frac{1}{2}}{}'(x)$ in place of $\eta_{s-\frac{1}{2}}(x)$ and $\zeta_{s-\frac{1}{2}}(x)$, and consequently that the residue at $s=\frac{1}{2}$ which arises in the equation corresponding to (3·2) gives a term $\cos x \,.\, e^{ix} \,.\, \cot\frac{1}{2}\theta$.

Thus

$$\mathrm{A} = -\tfrac{1}{2}\tan\tfrac{1}{2}\theta + \tfrac{1}{2}e^{2ix}\cot\tfrac{1}{2}\theta + 2\pi i \sum_{\nu} \nu/(\nu^2-\tfrac{1}{4}) \,.\, \eta_{\nu-\frac{1}{2}}{}'(x)/[\partial/\partial s\, \zeta_{s-\frac{1}{2}}{}'(x)]_{s=\nu}\,.$$

$$\left[\frac{e^{[\nu(\pi-\theta)-\frac{1}{4}\pi]i}}{e^{2\nu\pi i}+1}\mathrm{C}_1 - \frac{e^{-[\nu(\pi-\theta)-\frac{1}{4}\pi]i}}{e^{-2\nu\pi i}+1}\mathrm{C}_2\right] - \int_{\mathrm{P}}^{\mathrm{Q}} \frac{s}{s^2-\frac{1}{4}}\, e^{-[s(\pi+\theta)+\frac{1}{4}\pi]i}\,\mathrm{C}_1\, ds$$

$$-\int_{\mathrm{S}}^{\mathrm{T}} \frac{s}{s^2-\frac{1}{4}}\, e^{[s(\pi-\theta)-\frac{1}{4}\pi\ i}\,\mathrm{C}_1\, ds + \int_{\mathrm{C}} \frac{s}{s^2-\frac{1}{4}}\,\frac{\partial/\partial x\,\{x^{\frac{1}{2}}\mathrm{S}_s{}^{(1)}(x)\}}{\partial/\partial x\,\{x^{\frac{1}{2}}\mathrm{S}_s{}^{(2)}(x)\}}\, e^{[s(\pi-\theta)-\pi]i}\,\mathrm{C}_1\, ds. \quad (6{\cdot}2)$$

(7) The next step is to investigate the co-ordinates of the points Q and S, with a view to determining the magnitudes of the terms in equations (6·1) and (6·2).

By actual plotting of the curves it is found that the imaginary co-ordinate of Q lies between $0{\cdot}6\,x$ and $1{\cdot}5\,x$, while the imaginary co-ordinate of S lies between $0{\cdot}6x$ and 0.

Hence when x is very large, it is seen that the integral from P to Q involves an exponential, the real part of whose index, $(\pi+\theta)$ times the imaginary co-ordinate of Q, is large and negative. This integral is thus very small. The same is true of the integral from S to T, except when θ is near to π.

In this latter case, S is given very approximately by

$$x\left(1-\frac{1}{24}\phi^2 + i\frac{\sqrt{3}}{24}\phi^2\right), \text{ where } \phi = \pi-\theta.$$

The index of the exponential involved is thus $-\dfrac{\sqrt{3}}{24}\,.\,(\pi-\theta)^3$, and thus the integral is no longer very small when $(\pi-\theta)$ is of the order $x^{-\frac{1}{3}}$.

(8) We have next to consider the terms in A and B (equations 6·1, 6·2) arising from the zeros of $\zeta_{s-\frac{1}{2}}{}'(x)$ and $\zeta_{s-\frac{1}{2}}(x)$ respectively.

For $\zeta_{s-\frac{1}{2}}{}'(x)$ the work is given by Watson,* whose results are as follow.

* Watson, 'Roy. Soc. Proc.,' A, vol. 95, pp. 90–97 (1918).

The large zeros of $\zeta_{s-\frac{1}{2}}'(x)$ are given approximately by the equation

$$x(\sinh\gamma-\gamma\cosh\gamma)-\tfrac{1}{4}\pi i = -(m+\tfrac{1}{2})\pi i,$$

where m is an integer large compared with x, and the corresponding value (ν_m) of ν which makes $\zeta_{\nu-\frac{1}{2}}'(x)$ have a simple zero is

$$\nu_m \sim (m+\tfrac{1}{4})\pi^2/2(\log m)^2-(m+\tfrac{1}{4})\pi i/\log m.$$

The corresponding value of γ is

$$\gamma_m \sim -\log m+\tfrac{1}{2}\pi i.$$

These zeros lie in Watson's region 7, in which

$$\eta_{s-\frac{1}{2}}(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}}\{S_s^{(1)}(x)-e^{-2s\pi i}S_s^{(2)}(x)\}/(1-e^{-2s\pi i}),$$

$$\zeta_{s-\frac{1}{2}}(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}}\{S_s^{(2)}(x)-S_s^{(1)}(x)\}/(1-e^{-2s\pi i}).$$

Using the known approximations for $S_s^{(1)}(x)$ and $S_s^{(2)}(x)$ when $|\gamma|$ is not small, we find that at the zero ν_m

$$\eta_{\nu-\frac{1}{2}}'(x)/[\partial/\partial s\,\zeta_{s-\frac{1}{2}}'(x)]_{s=\nu} \sim 1/2\gamma_m \sim -1/2\log m.$$

Hence since the imaginary part of ν_m is negative, the term in A arising from a large zero of $\zeta_{s-\frac{1}{2}}'(x)$ is approximately

$$-\frac{\pi i}{\nu_m \log m}\left[e^{-(m+\frac{1}{4})\pi(\pi+\theta)/\log m}\,C_1'-e^{-(m+\frac{1}{4})\pi(\pi-\theta)/\log m}\,C_2'\right],$$

which is very small as long as θ is not nearly equal to π.

For the small zeros of $\zeta_{s-\frac{1}{2}}'(x)$, Watson* obtains the approximate values

$$\nu = x+\rho x^{\frac{1}{3}}e^{-\frac{1}{3}\pi i},$$

where for the first three

$$\rho = 0{\cdot}8083,\ 2{\cdot}577,\ 3{\cdot}83.$$

Using these results, we arrive at the conclusion that the first terms in the series in equation (6·2) contain exponentials with indices

$$-\tfrac{1}{2}\sqrt{3}\rho\,.\,(\pi+\theta)x^{\frac{1}{3}} \text{ and } -\tfrac{1}{2}\sqrt{3}\rho\,(\pi-\theta)\,x^{\frac{1}{3}}$$

and are thus again small, for large values of x, except when θ is nearly equal to π.

The work for the corresponding terms in B (equation 6·1) is very similar and gives similar results. As these terms are negligible, it is unnecessary to put down the details of the calculations.

(9) We have shown that for θ not near to π (more accurately, as long as $(\pi-\theta)x^{\frac{1}{3}}$ is large), we have approximately, neglecting terms containing exponentials with large negative indices,

$$A = -\tfrac{1}{2}\tan\tfrac{1}{2}\theta+\tfrac{1}{2}e^{2ix}\cot\tfrac{1}{2}\theta+\int_C \frac{s}{s^2-\frac{1}{4}}\,\frac{\partial/\partial x\,\{x^{\frac{1}{2}}S_s^{(1)}(x)\}}{\partial/\partial x\,\{x^{\frac{1}{2}}S_s^{(2)}(x)\}}\,e^{[s(\pi-\theta)-\frac{1}{2}\pi]i}\,C_1\,ds, \tag{9·1}$$

$$B = -\tfrac{1}{2}\tan\tfrac{1}{2}\theta-\tfrac{1}{2}e^{2ix}\cot\tfrac{1}{2}\theta+\int_C \frac{s}{s^2-\frac{1}{4}}\,\frac{S_s^{(1)}(x)}{S_s^{(2)}(x)}\,e^{[s(\pi-\theta)-\frac{1}{2}\pi]i}\,C_1\,ds. \tag{9·2}$$

* Watson, *loc. cit.*, p. 97.

It remains to approximate to the integrals along the curve C. We must first write down the asymptotic expression for C_1. For the corresponding part of $P_{s-\frac{1}{2}}(-\cos\theta)$ we have, $|s|$ being large,

$$s^{[s(\pi-\theta)-\frac{1}{4}\pi]i}(2\pi s\sin\theta)^{-\frac{1}{2}}\left(1+\frac{i}{8s}\cot\theta\right).$$

Differentiating, we get

$$C_1 \sim -i(s/2\pi\sin\theta)^{\frac{1}{2}}\left(1-\frac{3i}{8s}\cot\theta\right). \tag{9·3}$$

The asymptotic expansions for $S_s^{(1)}(x)$ and $S_s^{(2)}(x)$ are given by Debye*; he, however, uses $i\tau_0$ where we have γ.

So long as $|\gamma|$ is not small, which is true along the curve C when θ is not near to π, we have

$$S_s^{(1)}(x) \sim \exp\{x(\sinh\gamma-\gamma\cosh\gamma)-\tfrac{1}{4}\pi i\}\{\tfrac{1}{2}\pi x\sin(-i\gamma)\}^{-\frac{1}{2}}$$
$$\left[1+\left(\frac{1}{8}-\frac{5}{24}\coth^2\gamma\right)\Big/x\sinh\gamma\right],$$

$$S_s^{(2)}(x) \sim \exp\{-x(\sinh\gamma-\gamma\cosh\gamma)+\tfrac{1}{4}\pi i\}\{\tfrac{1}{2}\pi x\sin(-i\gamma)\}^{-\frac{1}{2}}$$
$$\left[1-\left(\frac{1}{8}-\frac{5}{24}\coth^2\gamma\right)\Big/x\sinh\gamma\right].$$

After some reduction we get for the integral of equation (9·1), neglecting terms of relative order x^{-2},

$$(2\pi\sin\theta)^{-\frac{1}{2}}\int_C s^{-\frac{1}{2}}\exp\{2x(\sinh\gamma-\gamma\cosh\gamma)+ix\cosh\gamma(\pi-\theta)-\tfrac{1}{4}\pi i\}$$
$$\left[1-\frac{3}{8}\frac{i\cot\theta}{x\cosh\gamma}+\frac{1/4+7/12\coth^2\gamma}{x\sinh\gamma}\right]ds \tag{9·4}$$

and for equation (9·2),

$$-(2\pi\sin\theta)^{-\frac{1}{2}}\int_C s^{-\frac{1}{2}}\exp\{2x(\sinh\gamma-\gamma\cosh\gamma)+ix\cosh\gamma(\pi-\theta)-\tfrac{1}{4}\pi i\}$$
$$\left[1-\frac{3}{8}\frac{i\cot\theta}{x\cosh\gamma}+\frac{1/4-5/12\coth^2\gamma}{x\sinh\gamma}\right]ds. \tag{9·5}$$

The curve C is, as we anticipated, a "path of steepest descent" for our integrals, the "saddle-point" being at $\gamma=\frac{1}{2}i(\pi-\theta)$.

To evaluate them, we put

$$2\sinh\gamma-2\gamma\cosh\gamma+i(\pi-\theta)\cosh\gamma = 2i\cos\tfrac{1}{2}\theta-\tau^2.$$

Along C, τ^2 is real and positive, and the integrals involve $e^{-x\tau^2}$ multiplied

* Debye, 'Math. Annalen,' vol. 67, p. 535 (1909); 'München Sitzungsber.,' Abh. 5 (1910).

by a factor which may be expanded in positive integral powers of τ. The individual terms, of the type

$$\int_{-\infty}^{\infty} \exp(-x\tau^2)\,\tau^n\,d\tau,$$

are known integrals, and thus the evaluation may be effected to any term in the asymptotic series, though it becomes extremely laborious even to include terms of relative order x^{-2} (for which, of course, additional terms are required in the approximations, both for the Legendre and the Bessel functions).

The final results obtained are

$$\begin{aligned} A = {} & -\tfrac{1}{2}\tan\tfrac{1}{2}\theta+\tfrac{1}{2}e^{2ix}\cot\tfrac{1}{2}\theta+\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}\theta\exp(2ix\cos\tfrac{1}{2}\theta) \\ & \{1+\tfrac{1}{2}i\sin^2\tfrac{1}{2}\theta\sec^3\tfrac{1}{2}\theta x^{-1} \\ & -\tfrac{1}{4}(5\tan^2\tfrac{1}{2}\theta+7\tan^4\tfrac{1}{2}\theta)\sec^2\tfrac{1}{2}\theta x^{-2}\}, \end{aligned} \qquad (9\cdot6)$$

$$\begin{aligned} B = {} & -\tfrac{1}{2}\tan\tfrac{1}{2}\theta-\tfrac{1}{2}e^{2ix}\cot\tfrac{1}{2}\theta-\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}\theta\exp(2ix\cos\tfrac{1}{2}\theta) \\ & \{1-\tfrac{1}{2}i\sin^2\tfrac{1}{2}\theta\sec^3\tfrac{1}{2}\theta x^{-1} \\ & +\tfrac{1}{4}(5\tan^2\tfrac{1}{2}\theta+7\tan^4\tfrac{1}{2}\theta)\sec^2\tfrac{1}{2}\theta x^{-2}\}. \end{aligned} \qquad (9\cdot7)$$

Now
$$Y_1+iY_2 = i\left[\frac{\partial A}{\partial\theta}-\frac{B}{\sin\theta}\right],$$

$$Z_1+iZ_2 = i\left[\frac{\partial B}{\partial\theta}-\frac{A}{\sin\theta}\right].$$

On differentiating, it is found that the terms arising from the first two terms in equations (9·6), (9·7) vanish.

The values of the quantities that we are investigating are thus approximately, for θ not near to π,

$$Y_1+iY_2=\tfrac{1}{2}x\exp(2ix\cos\tfrac{1}{2}\theta)\{1+\tfrac{1}{2}ix^{-1}\sec^3\tfrac{1}{2}\theta-\tfrac{7}{4}x^{-2}\sin^2\tfrac{1}{2}\theta\sec^6\tfrac{1}{2}\theta\},$$

$$Z_1+iZ_2=-\tfrac{1}{2}x\exp(2ix\cos\tfrac{1}{2}\theta)\{1+\tfrac{1}{2}ix^{-1}\cos\theta\sec^3\tfrac{1}{2}\theta+\tfrac{7}{4}x^{-2}\sin^2\tfrac{1}{2}\theta\sec^6\tfrac{1}{2}\theta\}.$$

The following Table shows the values of Y_1, Y_2, Z_1, Z_2 calculated for $ka = 10$ from the second approximation above. These are denoted by the letter A; and for comparison the corresponding values obtained by Proudman, Doodson, and Kennedy are also given under the letter B.

$ka = 10.$

θ.	Y_1.		Y_2.		Z_1.		Z_2.	
	A.	B.	A.	B.	A.	B.	A.	B.
°								
0	1·76	2·02	4·67	4·38	−1·76	−2·02	−4·67	−4·38
10	2·16	2·21	4·52	4·57	−2·16	−2·32	−4·52	−4·36
20	3·12	2·90	3·92	4·16	−3·13	−3·16	−3·91	−3·93
30	4·34	4·26	2·51	2·30	−4·36	−4·31	−2·47	−2·51
45	4·78	4·98	−1·52	−1·35	−4·76	−4·79	1·61	1·64
60	0·60	0·20	−4·98	−5·09	−0·40	−0·40	4·99	4·98
70	−3·62	−3·27	−3·48	−3·83	3·81	3·85	3·25	3·34
80	−4·84	−4·41	1·37	1·96	4·67	4·72	−1·80	−1·75
90	−0·73	−1·40	5·00	5·09	0·02	0·14	−5·00	−5·19
100	4·53	4·51	2·33	0·97	−4·84	−4·92	−1·28	−1·52
110	3·47	4·22	−3·84	−3·27	−1·89	−2·45	4·65	4·81

(10) As a preliminary to considering values of θ near to π, let us take the case in which $\theta = \pi$.

In this case we must differentiate our series before evaluating them. We have from § 1 :—

$$Y = -\cos\phi \frac{e^{-ikr}}{kr} \sum_1^\infty (-1)^n \frac{2n+1}{n(n+1)} \left[\frac{S_n'}{E_n'} \frac{\partial}{\partial\theta}\{\sin\theta\, P_n'(\cos\theta)\} - \frac{S_n}{E_n} P_n'(\cos\theta)\right],$$

$$Z = -\sin\phi \frac{e^{-ikr}}{kr} \sum_1^\infty (-1)^n \frac{2n+1}{n(n+1)} \left[\frac{S_n}{E_n} \frac{\partial}{\partial\theta}\{\sin\theta\, P_n'(\cos\theta)\} - \frac{S_n'}{E_n'} P_n'(\cos\theta)\right]. \quad (10\cdot1)$$

Now $\dfrac{\partial}{\partial\theta}\{\sin\theta\, P_n'(\cos\theta)\} = n(n+1)\, P_n(\cos\theta) - \cos\theta\, P_n'(\cos\theta).$

Also, when $\theta = \pi$, $(-1)^n P_n(\cos\theta) = 1$; $(-1)^n P_n'(\cos\theta) = -\frac{1}{2}n(n+1)$.
Hence, for $\theta = \pi$,

$$Y = -\cos\phi \frac{e^{-ikr}}{kr} \tfrac{1}{2}(A+B), \qquad Z = -\sin\phi \frac{e^{-ikr}}{kr} \tfrac{1}{2}(A+B), \quad (10\cdot2)$$

where A and B denote the series

$$A = \sum_1^\infty (2n+1) \frac{S_n(ka)}{E_n(ka)}; \qquad B = \sum_1^\infty (2n+1) \frac{S_n'(ka)}{E_n'(ka)}. \quad (10\cdot3)$$

To the approximate evaluation of these series we now proceed.

In Watson's notation

$$A = i\sum_1^\infty (2n+1)\frac{\psi_n(x)}{\zeta_n(x)}; \qquad B = i\sum_1^\infty (2n+1)\frac{\psi_n'(x)}{\zeta_n'(x)}.$$

By work which is exactly parallel to that of paragraphs 3, 4,

$$A = -\sin x e^{ix} - 2\pi\sum_\nu \frac{\nu\eta_{\nu-\frac{1}{2}}(x)}{(e^{2\nu\pi i}+1)\,[\partial/\partial s\,\zeta_{s-\frac{1}{2}}(x)]_{s=\nu}} + i\int_{C'} s\left[\frac{\eta_{s-\frac{1}{2}}(x)}{\zeta_{s-\frac{1}{2}}(x)} + \frac{1}{e^{2s\pi i}+1}\right] ds,$$

$$B = -i\cos x e^{ix} - 2\pi\sum_\nu \frac{\nu\eta_{\nu-\frac{1}{2}}'(x)}{(e^{2\nu\pi i}+1)\,[\partial/\partial s\,\zeta_{s-\frac{1}{2}}'(x)]_{s=\nu}} + i\int_{C'} s\left[\frac{\eta'_{s-\frac{1}{2}}(x)}{\zeta'_{s-\frac{1}{2}}(x)} + \frac{1}{e^{2\nu\pi i}+1}\right] ds. \tag{10·4}$$

The series in (10·4) gives a negligible contribution when x is large, each term involving an exponential with large negative index.

As regards the contour integral, we may note that in this case, the curve C as defined in §2 degenerates into the boundary between regions 5 and 7, the boundary between regions 1 and 3, and the real axis from x to $+\infty$.

The expression which we shall have to integrate over C is, for A, $\int s\frac{S_s^{(1)}(x)}{S_s^{(2)}(x)}ds$.

The remainder, by a process similar to that of §5, is seen to be

$$-\int_P^Q se^{-2s\pi i}\,ds + \int_P^O \frac{s}{e^{2s\pi i}+1}\,ds - \int_S^T \frac{s}{e^{-2s\pi i}+1}\,ds + \int_O^S \frac{s}{e^{2s\pi i}+1}\,ds$$

$$= -\left[\frac{ise^{-2s\pi i}}{2\pi} + \frac{1}{4\pi^2}e^{-2s\pi i}\right]_Q - 2\int_O^T \frac{s}{e^{-2s\pi i}+1}\,ds + \int_O^S s\,ds$$

$$= \text{a negligible quantity} + \frac{1}{24} + \tfrac{1}{2}x^2,$$

since S is the point x, and

$$\int_O^T \frac{s}{e^{-2s\pi i}+1}\,ds = -\int_0^\infty \frac{y\,dy}{1+e^{2y\pi}} = -\frac{1}{4\pi^2}\left\{1-\frac{1}{2^2}+\frac{1}{3^2}-\dots\right\} = -\frac{1}{48}.$$

We have, finally, to approximate to $\int_C s\frac{S_s^{(1)}(x)}{S_s^{(2)}(x)}ds$.

Along the boundary between regions 5 and 7, $|\gamma|$ is not small, and we have the approximation

$$\frac{S_s^{(1)}(x)}{S_s^{(2)}(x)} = \exp\{2x(\sinh\gamma - \gamma\cosh\gamma) - \tfrac{1}{2}\pi i\}.$$

Since $\sinh\gamma - \gamma\cosh\gamma$ is real and negative on this boundary, the contribution from this part of C involves an exponential with a large negative index and may therefore be neglected.

In region 1, approximations valid right up to the point x are :*

$$S_s^{(1)}(x) = -\tfrac{2}{3}i \tanh\gamma \, \exp\{x(\sinh\gamma-\gamma\cosh\gamma)+\eta\} \\ \{e^{1/6\pi i}I_{-1/3}(\eta)+e^{-1/6\pi i}I_{1/3}(\eta)\},$$

$$S_s^{(2)}(x) = \tfrac{2}{3}i \tanh\gamma \exp\{-(x\sinh\gamma-\gamma\cosh\gamma)-\eta\} \\ \{e^{-1/6\pi i}I_{-1/3}(\eta)+e^{1/6\pi i}I_{1/3}(\eta)\},$$

where $\eta = \frac{1}{3}x\sinh^3\gamma\,\mathrm{sech}^2\gamma$, and the phase of η is to vanish when γ is real and positive. I is the modified Bessel function.

Thus the contribution from this part may be taken as

$$-\int s\exp\{2x(\sinh\gamma-\gamma\cosh\gamma)+2\eta\}\frac{e^{1/6\pi i}I_{-1/3}(\eta)+e^{-1/6\pi i}I_{1/3}(\eta)}{e^{-1/6\pi i}I_{-1/3}(\eta)+e^{1/6\pi i}I_{1/3}(\eta)}\,ds.$$

The principal part of this integral, of course, arises for $|\gamma|$ small. We now express everything in terms of η.

After some reduction, the term of the integral involving the highest power of x is found to be

$$-\int 3^{-1/3}x^{4/3}\eta^{-1/3}\frac{e^{1/6\pi i}I_{-1/3}(\eta)+e^{-1/6\pi i}I_{1/3}(\eta)}{e^{-1/6\pi i}I_{-1/3}(\eta)+e^{1/6\pi i}I_{1/3}(\eta)}\,d\eta.$$

Now the initial phase of γ (at x) is $\frac{2}{3}\pi$, and hence the initial phase of η is 2π, and thus, writing $\eta = te^{2\pi i}$, t being a real positive quantity, the approximation to this part of the integral is

$$3^{-1/3}x^{4/3}e^{-1/3\pi i}\int_0^\infty t^{-1/3}\frac{I_{-1/3}(t)-I_{1/3}(t)}{I_{-1/3}(t)+e^{-1/3\pi i}I_{1/3}(t)}\,dt.$$

In an exactly similar manner, the approximation to the part arising from region 2 is found to be

$$3^{-1/3}x^{4/3}e^{-1/3\pi i}\int_0^\infty t^{-1/3}\frac{I_{-1/3}(t)-I_{1/3}(t)}{I_{-1/3}(t)+e^{1/3\pi i}I_{1/3}(t)}\,dt.$$

Adding, the final value of $\int_C s\frac{S_s^{(1)}(x)}{S_s^{(2)}(x)}\,ds$ is

$$3^{-1/3}x^{4/3}e^{-1/3\pi i}\int_0^\infty t^{-1/3}\frac{\{I_{-1/3}(t)-I_{1/3}(t)\}\{2I_{-1/3}(t)+I_{1/3}(t)\}}{I_{-1/3}^2(t)+I_{-1/3}(t)I_{1/3}(t)-I_{1/3}^2(t)}\,dt \\ = 3^{-1/3}x^{4/3}e^{-1/3\pi i}\,P, \text{ say.}$$

In the same way the approximation to the corresponding integral in B, $\int_C s\frac{\partial/\partial x\{x^{1/2}S_s^{(1)}(x)\}}{\partial/\partial x\{x^{1/2}S_s^{(2)}(x)\}}\,ds$, is found to be

$$-3^{-1/3}x^{4/3}e^{-1/3\pi i}\int_0^\infty t^{-1/3}\frac{\{I_{-2/3}(t)-I_{2/3}(t)\}\{I_{-2/3}(t)+2I_{2/3}(t)\}}{I_{-2/3}^2(t)+I_{-2/3}(t)I_{2/3}(t)+I_{2/3}^2(t)}\,dt, \\ = -3^{-1/3}x^{4/3}e^{-1/3\pi i}\,Q, \text{ say.}$$

* These approximations are not given explicitly by Watson, but can be obtained by application of the methods of his papers (*loc. cit.*, § 8, and 'Proc. Camb. Phil. Soc.,' vol. 19, p. 96 (1916)).

The two integrals P and Q are definite numbers which can be calculated, to a sufficient approximation, by actually plotting the functions. For this purpose the tables of the modified Bessel functions of orders $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, as given by Dinnik,* are of use, but additional figures, for small values of the argument, are also needed. These have been calculated directly from the series. For extremely small values of t, and for large values of t, the Bessel functions may be replaced by appropriate approximations, and thus it is only for intermediate values of t, say from $t = 0{\cdot}02$ to $t = 2{\cdot}2$, that the curves need be plotted.

The final results are

$$P = 1{\cdot}39, \qquad Q = 1{\cdot}24,$$

which are probably accurate to the second place of decimals.

(11) Collecting together our results, we have for the leading terms

$$A = \tfrac{1}{2}ix^2 + 1{\cdot}39 \times 3^{-1/3}\, e^{1/6\pi i}\, x^{4/3},$$

$$B = \tfrac{1}{2}ix^2 - 1{\cdot}24 \times 3^{-1/3}\, e^{1/6\pi i}\, x^{4/3}.$$

It is useful at this stage to compare the results obtained with those arrived at by direct arithmetical evaluation. In the paper of Messrs. Proudman, Doodson, and Kennedy, to which reference has already been made, the various series involved are calculated for $x = ka = 10$. It is possible to pick out what corresponds to the above series. In fact, the series A above is the same as the $C + iD$ of equations (6), 2 of the paper quoted, and B is $C' + iD'$.

From Tables V–VIII, we conclude that for $x = 10$,

$$A = 19{\cdot}58 + i59{\cdot}64, \qquad B = -21{\cdot}00 + i43{\cdot}48.$$

The approximations above give

$$A = 17{\cdot}98 + i60{\cdot}38, \qquad B = -16{\cdot}04 + i40{\cdot}76,$$

which is as close agreement, 10 not being very large compared with unity, as can be expected.

When, however, we come to take the sum of A and B, the terms in $x^{4/3}$, being very nearly equal and opposite, practically cancel, and the terms arising from the next approximation, involving presumably $x^{2/3}$, will become important. We find

$$\tfrac{1}{2}(A + B) = \tfrac{1}{2}ix^2 + 0{\cdot}08 + 3^{-1/3}\, e^{1/6\pi i}\, x^{4/3},$$

* A. Dinnik, 'Archiv der Math. u. Physik,' (3), vol. 22, p. 226 (1914). Some misprints may be noted. The correct values of $I_{-1/3}(x)$, $I_{-2/3}(x)$ are as follow :—

x.	$I_{-1/3}(x)$.	x	$I_{-2/3}(x)$.
2·2	2·5626	2·8	3·7595
3·0	4·7754	8·0	415·01
6·0	66·55		

and, in the notation of the first paragraph,

$$Y_1 = Z_1 = -0{\cdot}045\,(ka)^{4/3},$$

$$Y_2 = Z_2 = -\tfrac{1}{2}(ka)^2 - 0{\cdot}026\,(ka)^{4/3}.$$

For $ka = 10$, these give

$$Y_1 = Z_1 = -0{\cdot}97\,; \qquad Y_2 = Z_2 = -50{\cdot}57,$$

whereas the calculated values are

$$Y_1 = Z_1 = +0{\cdot}71\,; \qquad Y_2 = Z_2 = -51{\cdot}56.$$

It would thus be desirable to carry the approximations to our contour integrals at least one stage further, but the labour involved seems to be prohibitive.

(12) There remains to discuss the case of θ nearly equal to π. It is more convenient to differentiate the series before transforming them.

We get
$$Y = -\cos\phi\,\frac{e^{-ikr}}{kr}\,[\Sigma_2 - \cos\theta\,\Sigma_4 - \Sigma_3],$$

$$Z = -\sin\phi\,\frac{e^{-ikr}}{kr}\,[\Sigma_1 - \cos\theta\,\Sigma_3 - \Sigma_4],$$

where
$$\Sigma_1 = \sum_1^\infty (-1)^n (2n+1)\,P_n(\cos\theta)\,\frac{S_n(ka)}{E_n(ka)},$$

$$\Sigma_2 = \sum_1^\infty (-1)^n (2n+1)\,P_n(\cos\theta)\,\frac{S_n'(ka)}{E_n'(ka)},$$

$$\Sigma_3 = \sum_1^\infty (-1)^n\,\frac{2n+1}{n(n+1)}\,P_n'(\cos\theta)\,\frac{S_n(ka)}{E_n(ka)},$$

$$\Sigma_4 = \sum_1^\infty (-1)^n\,\frac{2n+1}{n(n+1)}\,P_n'(\cos\theta)\,\frac{S_n'(ka)}{E_n'(ka)}.$$

The work of transformation is exactly parallel to that of the previous cases.

For Σ_1 we have finally to approximate to

$$\int s\,P_{s-\frac{1}{2}}(-\cos\theta)\,\frac{S_s^{(1)}(x)}{S_s^{(2)}(x)}\,ds$$

taken over the same curve C as in the case of $\theta = \pi$.

The remainder is

$$-\int_P^Q s\,P_{s-\frac{1}{2}}(-\cos\theta)\,e^{-2s\pi i}\,ds - 2\int_O^T s\,P_{s-\frac{1}{2}}(-\cos\theta)\,\frac{1}{e^{-2s\pi i}+1}\,ds$$
$$+\int_O^S s\,P_{s-\frac{1}{2}}(-\cos\theta)\,ds.$$

Of these the first may be neglected. Also S is the point $s = x$.

Since θ is near to π, we may replace the Legendre function in the third integral by $J_0(2s\cos\frac{1}{2}\theta)$, and the integral becomes

$$\int_0^K s\,J_0(2s\cos\tfrac{1}{2}\theta)\,ds = \tfrac{1}{2}x\sec\tfrac{1}{2}\theta\,J_1(2x\cos\tfrac{1}{2}\theta),$$

which tends to $\frac{1}{2}x^2$ as θ tends to π.

Also the important part of the second integral is from $|s|$ small, and so, swinging the contour round into the imaginary axis, we get

$$2\int_0^\infty \frac{t\,I_0(2t\cos\frac{1}{2}\theta)}{1+e^{2t\pi}}\,dt.$$

This can be evaluated as a series in $\cos^2\frac{1}{2}\theta$, the coefficients involving Bernoulli's numbers. The first terms are

$$\frac{1}{24}+\frac{17}{960}\cos^2\tfrac{1}{2}\theta+\ldots.$$

For the integral over C the process is practically as in paragraph 10, save that we have an additional factor $J_0(2s\cos\frac{1}{2}\theta)$.

Thus the result will be

$$0{\cdot}964\,J_0(2x\cos\tfrac{1}{2}\theta)\,e^{1/6\pi i}\,x^{4/3}.$$

Altogether the leading terms of Σ_1, Σ_2 are

$$\Sigma_1 = \tfrac{1}{2}ix\sec\tfrac{1}{2}\theta\,J_1(2x\cos\tfrac{1}{2}\theta)+0{\cdot}964\,J_0(2x\cos\tfrac{1}{2}\theta)\,e^{1/6\pi i}\,x^{4/3},$$

$$\Sigma_2 = \tfrac{1}{2}ix\sec\tfrac{1}{2}\theta\,J_1(2x\cos\tfrac{1}{2}\theta)-0{\cdot}86\,J_0(2x\cos\tfrac{1}{2}\theta)\,e^{1/6\pi i}\,x^{4/3}.$$

In the series Σ_3 the extra integrals are

$$-2\int_0^T \frac{s}{s^2-\frac{1}{4}}\,P_{s-\frac{1}{2}}'(-\cos\theta)\,\frac{1}{e^{-2s\pi i}+1}\,ds+\int_0^K \frac{s}{s^2-\frac{1}{4}}\,P_{s-\frac{1}{2}}'(-\cos\theta)\,ds,$$

of which the first gives

$$\frac{1}{48\sin^2\frac{1}{2}\theta}+\frac{1}{160}\cos^2\tfrac{1}{2}\theta+\text{terms of higher degree in }\cos\tfrac{1}{2}\theta.$$

The second integral can easily be shown to be approximately

$$\operatorname{cosec}^2\theta\{1-J_0(2x\cos\tfrac{1}{2}\theta)\}.$$

The integral $\int \frac{s}{s^2-\frac{1}{4}}\,P_{s-\frac{1}{2}}'(-\cos\theta)\,\frac{S_s^{(1)}(x)}{S_s^{(2)}(x)}\,ds$ gives, on putting for $P_{s-\frac{1}{2}}'(-\cos\theta)$ the approximate value $\frac{1}{2}s\sec\frac{1}{2}\theta\operatorname{cosec}^2\frac{1}{2}\theta\,J_1(2s\cos\frac{1}{2}\theta)$, the expression $\frac{1}{2x}\operatorname{cosec}^2\frac{1}{2}\theta\sec\frac{1}{2}\theta\,J_1(2x\cos\frac{1}{2}\theta)\times 0{\cdot}964\,e^{-1/3\pi i}\,x^{4/3}$.

Thus for the leading terms

$$\Sigma_3 = -i\operatorname{cosec}^2\theta\{1-J_0(2x\cos\tfrac{1}{2}\theta)\}$$
$$-0{\cdot}482\operatorname{cosec}^2\tfrac{1}{2}\theta\sec\tfrac{1}{2}\theta\,J_1(2x\cos\tfrac{1}{2}\theta)\,e^{1/6\pi i}\,x^{1/3},$$

and in a similar way

$$\Sigma_4 = -i \operatorname{cosec}^2 \theta \{1 - J_0(2x \cos \tfrac{1}{2}\theta)\}$$
$$+ 0{\cdot}43 \operatorname{cosec}^2 \tfrac{1}{2}\theta \sec \tfrac{1}{2}\theta J_1(2x \cos \tfrac{1}{2}\theta) e^{1/6\pi i} x^{1/3}.$$

To compare with the figures for $x = 10$, we note that

$$\Sigma_1 = C + iD, \quad \Sigma_2 = C' + iD', \quad \Sigma_3 = A + iB, \quad \Sigma_4 = A' + iB',$$

in the notation of Proudman's paper.

The calculated values of the series give, for $\theta = 170°$,

$$\Sigma_1 = 5{\cdot}65 + 37{\cdot}47i, \quad \Sigma_2 = -5{\cdot}12 + 28{\cdot}84i, \quad \Sigma_3 = -7{\cdot}08 - 24{\cdot}20i,$$
$$\Sigma_4 = 6{\cdot}25 - 18{\cdot}04i,$$

while the above approximations give

$$\Sigma_1 = 6{\cdot}7 + 37{\cdot}2i, \quad \Sigma_2 = -5{\cdot}9 + 29{\cdot}9i, \quad \Sigma_3 = -6{\cdot}03 - 24{\cdot}3i,$$
$$\Sigma_4 = 5{\cdot}38 - 17{\cdot}7i.$$

The following Table shows the values of the series Σ_1, Σ_2, Σ_3, Σ_4, calculated from the above approximations for $ka = 10$ in the range of values of θ from 170° to 180°. The final figures for Y_1, Y_2, Z_1, Z_2 (denoted by A), given therefrom by the formulæ

$$Y_1 + iY_2 = \Sigma_3 - \cos\theta\, \Sigma_4 - \Sigma_2,$$
$$Z_1 + iZ_2 = \Sigma_4 + \cos\theta\, \Sigma_3 - \Sigma_1,$$

are also shown, and for comparison the values obtained of these by Proudman (denoted by B).

θ.	Σ_1.	Σ_2.	Σ_3.	Σ_4.
170	6·7 + 37·2i	−5·9 + 29·9i	−6·03 − 24·3i	5·38 − 17·7i
172	10·24 + 44·69i	−9·13 + 33·51i	−7·01 − 26·30i	6·25 − 18·64i
174	12·50 + 50·69i	−11·94 + 36·58i	−7·84 − 28·27i	6·99 − 19·39i
176	15·86 + 56·05i	−14·15 + 38·72i	−8·46 − 28·71i	7·55 − 19·89i
178	17·45 + 59·35i	−15·56 + 40·29i	−8·87 − 29·75i	7·91 − 20·06i
180	17·98 + 60·38i	−16·04 + 40·76i	$-\frac{1}{2}\Sigma_1$	$-\frac{1}{2}\Sigma_2$

θ.	Y_1. A.	Y_1. B.	Y_2. A.	Y_2. B.	Z_1. A.	Z_1. B.	Z_2. A.	Z_2. B.
170	−5·3	−7·12	−36·5	−35·28	4·5	6·59	−30·9	−31·67
172	−4·07	−4·76	−41·35	−41·0	2·95	4·7	−39·29	−38·0
174	−2·87	−2·33	−45·57	−45·6	2·29	3·0	−41·96	−43·9
176	−1·84	−0·23	−47·59	−49·3	0·13	1·6	−47 30	−48·4
178	−1·21	0·60	−49·99	−51·0	−0·68	0·8	−49·68	−51·0
180	−0·97	0·71	−50·57	−51·56	−0·97	0·71	−50·57	−51·56